

Name: \_\_\_\_\_

Key

Limit Definition of Derivative

1. Define derivative.

Slope of the tangent line

2. State the limit definition of a derivative.

$$\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

3. Given  $f(x)$ , find  $f'(x)$  by using the limit definition.

(a)  $f(x) = -4$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{-4 - (-4)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x} = 0$$

$$= 0$$

(b)  $f(x) = 5x + 1$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{5(x + \Delta x) + 1 - (5x + 1)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{5x + 5\Delta x + 1 - 5x - 1}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{5\Delta x}{\Delta x} = 5$$

(c)  $f(x) = -3x^2 + x + 5$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{-3(x + \Delta x)^2 + (x + \Delta x) + 5 - (-3x^2 + x + 5)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{-3(x^2 + 2x\Delta x + \Delta x^2) + x + \Delta x + 5 + 3x^2 - x - 5}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{-3x^2 - 6x\Delta x + 3\Delta x^2 + x + \Delta x + 5 + 3x^2 - x - 5}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{-6x\Delta x + 3\Delta x^2 + \Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} -6x + 3\Delta x + 1$$

(d)  $f(x) = x^3 + 2x$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^3 + 2(x + \Delta x) - (x^3 + 2x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{x^3 + x^2\Delta x + 2x^2\Delta x + 2x\Delta x^2 + x\Delta x^3 + \Delta x^3 + 2x - x^3 - 2x}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x^2 + 2x\Delta x + x\Delta x + \Delta x^2 + 2\cancel{\Delta x}}{\Delta x}$$

$$= 3x^2 + 2$$

$$\begin{aligned}
 (e) f(x) &= \sqrt{x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} (\sqrt{x+\Delta x} + \sqrt{x}) \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(x+\Delta x) - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}, \quad x \neq 0
 \end{aligned}$$

$$\begin{aligned}
 (f) f(x) &= \frac{2}{x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{\frac{2}{x+\Delta x} - \frac{2}{x}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2x - 2(x+\Delta x)}{x(x+\Delta x)} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{-2\Delta x}{x(x+\Delta x)} = \lim_{\Delta x \rightarrow 0} \frac{-2}{x(x+\Delta x)} = \boxed{-\frac{2}{x^2}}
 \end{aligned}$$

4. Using  $f(x) = -\frac{3}{2}x^2$ , predict if the slope of the tangent line will be positive or negative at  $x = -3$ ,  $x = 0$ , and  $x = 1$ . Then find the actual slope of the tangent line at these points.

$$\begin{aligned}
 &\text{@ } x = -3, \text{ the slope will be positive. } -\frac{3}{2}(-3) = 9 \\
 &\text{@ } x = 0, \text{ the slope will be } 0 \quad -\frac{3}{2}(0) = 0 \\
 &\text{@ } x = 1, \text{ the slope will be negative} \quad -\frac{3}{2}(1) = -\frac{3}{2} \\
 &\lim_{\Delta x \rightarrow 0} \frac{-\frac{3}{2}(x+\Delta x)^2 - \frac{3}{2}x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{-\frac{3}{2}(x^2 + 2x\Delta x + \Delta x^2) + \frac{3}{2}x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{-\frac{3}{2}\Delta x^2 - 3x\Delta x}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} -\frac{3}{2} \Delta x - 3x = \boxed{-3x}
 \end{aligned}$$

5. Given  $f(x) = x^2 + 2x + 1$ , find the slope of the tangent line at  $x = -3$ .

$$\begin{aligned}
 f'(-3) &= \lim_{x \rightarrow -3} \frac{(x^2 + 2x + 1) - (4)}{x - (-3)} = \lim_{x \rightarrow -3} \frac{x^2 + 2x - 3}{x + 3} \quad \boxed{f(-3) = 9 - 6 + 1 = 4} \\
 &= \lim_{x \rightarrow -3} \frac{(x+3)(x-1)}{x+3} = \lim_{x \rightarrow -3} (x-1) = -3 - 1 = \boxed{-4}
 \end{aligned}$$

6. Using the information from question #4, can you find the equation of the tangent line at  $x = -3$ ?

$$\begin{aligned}
 m &= 9 \quad (-3, -\frac{27}{2}) \quad -\frac{3}{2}(-3)^2 = -\frac{3}{2}(9) = -\frac{27}{2} \\
 y - (-\frac{27}{2}) &= 9(x - (-3)) \quad y - y_1 = m(x - x_1) \\
 y + \frac{27}{2} &= 9x + 27 \quad \boxed{y = 9x + 13.5} \\
 y &= 9x + 27 - \frac{27}{2} \\
 y &= 9x + 13.5
 \end{aligned}$$

ANSWERS: 1. slope of tan line 2.  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$  3a. 0 b. 5 c. -6x + 1 d.  $3x^2 + 2$  e.  $\frac{1}{2\sqrt{x}}$  f.  $-\frac{2}{x^2}$  g. 4, +, 0, -, 0, 0, -3 h. -4 i. 0 j.  $y = -4x - 16$