Calculus

Name____________________________

INVERSE FUNCTIONS

Seat # ________ Date __________________

Differential Equations: Growth and Decay

Problems 1-8: Differential Equations

In 1-3, given the initial condition, solve each differential equation by separating the variables. Express your final answers as $y = f(x)$.

1. $y' = \dfrac{5x}{y}$, with $f(1) = 6$

2. $y' = y\sqrt{x}$ with $f(0) = -3$

3. $y' = 100x - xy$ with $f(0) = 10$

In 4 and 5, write the differential equation that models the verbal statement. Then solve it by separating the variables.

4. The rate of change of h with respect to t is proportional to $10 - t$.

5. The rate of change of P with respect to x is inversely proportional to the square of x.

In 6 and 7, determine whether the proposed function is a solution for the given differential equation. Do NOT solve the differential equation, but rather verify the solution by plugging in the function and appropriate derivatives

6. Differential equation: $xy' - 2y = x^3 e^x$; proposed solution: $y = x^2 e^x$.

7. Differential equation: $y'' + y = 0$; proposed solution: $y = 7\cos x - 4\sin x$.

ESSAY FROM 1997 AB EXAM (you <u>may</u> use a graphing calculator for this question):

8. Let $v(t)$ be the velocity, in feet per second, of a skydiver at time t seconds, $t \geq 0$. After her parachute opens, her velocity satisfies the differential equation $\dfrac{dv}{dt} = -2v - 32$, with the initial condition $v(0) = -50$.

   a) Use separation of variables to find an expression for v in terms of t, where t is measured in seconds.
   b) Terminal velocity is defined as $\lim_{t \to \infty} v(t)$. Find the terminal velocity of the skydiver to the nearest foot per second.
   c) It is safe to land when her speed is 20 feet per second. At what time t does she reach this speed?

SEE OTHER SIDE

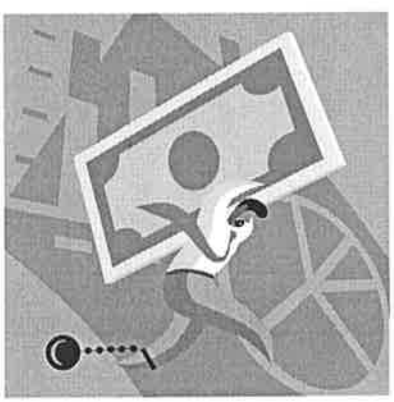
Problems 9-11: Growth and Decay

9. Earth's atmospheric pressure p is often modeled by assuming that the rate $\frac{dp}{dh}$ at which p changes with the altitude h is proportional to p. That is to say, $\frac{dp}{dh} = k \cdot p$. Suppose that the pressure at sea level ($h = 0$) is 1013 milibars and that the pressure at an altitude of 20 km is 90 milibars.

a) Write a solution to the differential equation shown above to express p, atmospheric pressure, as a function of h, altitude.
b) What is the atmospheric pressure at $h = 50$ km?
c) At what altitude does the pressure equal 900 milibars?

10. A colony of bacteria is grown under ideal conditions such that the instantaneous rate of change of bacteria is directly proportional to the number of bacteria present. Let b represent the bacteria present in the colony t hours after the experiment begins.

a) Write a differential equation that models the relationship.
b) Solve your differential equation. Suppose that at the end of 3 hours, there are 10,000 bacteria. At the end of 5 hours, there are 40,000 bacteria. How many bacteria were present initially?
c) When will the bacteria population reach one million?
d) What will the bacteria population be one full day after the experiment begins? Does this number make sense? Why?



11. One of the current goals of the federal government is to minimize the federal debt of the United States. In the last decades, the federal debt has grown following a pattern such that the rate of growth of the debt is directly proportional to the debt itself.

a) Write a differential equation that expresses this relationship. Let f represent the federal debt (in billions) and t the number of years.
b) In 1970 the federal debt was around \$370 billion. In 1980, the debt had grown to about \$900 billion. Solve your differential equation and write an equation to express the federal debt, f, as a function of t, the number of years since 1970.
c) Use your equation to estimate the debt for the year 2009. Compare your estimate to the actual debt for that year (\$11.5 trillion, or what is the same, \$11,500 billion.) How accurate is your equation?
d) Assuming the population of the United States was around 300 million people in 2009, what was your share of the federal debt?

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ANSWER KEY

INVERSE FUNCTIONS

Differential Equations: Growth and Decay

1. $y' = \frac{5x}{y} \Rightarrow \frac{dy}{dx} = \frac{5x}{y} \Rightarrow y \cdot dy = 5x \cdot dx \Rightarrow \frac{y^2}{2} = \frac{5x^2}{2} + C$ or $5x^2 - y^2 = A$
   with $f(1) = 6$: $5(1)^2 - (6)^2 = A$; so $A = -31$ and $y = \sqrt{5x^2 + 31}$

2. $y' = \sqrt{x} \cdot y \Rightarrow \frac{dy}{dx} = x^{\frac{1}{2}} y \Rightarrow \frac{1}{y} \cdot dy = x^{\frac{1}{2}} dx \Rightarrow \ln|y| = \frac{2}{3} x^{\frac{3}{2}} + C \Rightarrow y = A \cdot e^{\frac{2}{3}x^{\frac{3}{2}}}$
   with $f(0) = -3$: $-3 = A \cdot e^{\frac{2}{3}(0)^{\frac{3}{2}}}$; so $A = -3$ and $y = -3e^{\frac{2}{3}x^{\frac{3}{2}}}$

3. $y' = 100x - xy \Rightarrow \frac{dy}{dx} = x \cdot (100 - y) \Rightarrow \frac{1}{100 - y} \cdot dy = x \cdot dx \Rightarrow -\ln|100 - y| = \frac{1}{2} x^2 + C$
   So: $y = 100 + A \cdot e^{-x^2/2}$; with $f(0) = 10$: $10 = 100 + A \cdot e^{-(0)^2/2}$; so $A = -90$ and $y = 100 - 90e^{-x^2/2}$

4. $\frac{dh}{dt} = k \cdot (10 - t) \Rightarrow dh = k \cdot (10 - t) \cdot dt \Rightarrow h = k \cdot \left(10t - \frac{1}{2} t^2\right) + C$

5. $\frac{dP}{dx} = \frac{k}{x^2} \Rightarrow dP = k \cdot x^{-2} dx \Rightarrow P = -\frac{k}{x} + C$

6. $y = x^2 e^x \Rightarrow y' = 2xe^x + x^2 e^x$, plugging into the differential equation:
   $x(2xe^x + x^2 e^x) - 2(x^2 e^x) = x^3 e^x$
   So: $x^3 e^x = x^3 e^x$ and the proposed solution checks.

7. $y = 7\cos x - 4\sin x \Rightarrow y' = -7\sin x - 4\cos x \Rightarrow y'' = -7\cos x + 4\sin x$, plugging into the differential equation:
   $(-7\cos x + 4\sin x) + (7\cos x - 4\sin x) = 0$
   So: $0 = 0$ and the proposed solution checks.

8. (a) $\dfrac{dv}{dt} = -2v - 32 = -2(v+16)$

$\dfrac{dv}{v+16} = -2dt$

$\ln|v+16| = -2t + A$

$|v+16| = e^{-2t+A} = e^{A}e^{-2t}$

$v+16 = Ce^{-2t}$

$-50+16 = Ce^{0} \Rightarrow C = -34$

$v = -34e^{-2t} - 16$

(b) $\lim\limits_{t\to\infty} v(t) = \lim\limits_{t\to\infty}\left(-34e^{-2t} - 16\right) = -16$

(c) $v(t) = -34e^{-2t} - 16 = -20$

$e^{-2t} = \dfrac{2}{17} \Rightarrow t = -\dfrac{1}{2}\ln\left(\dfrac{2}{17}\right) \approx 1.070$

9. a) $\dfrac{dp}{dh} = k \cdot p \Rightarrow p = A \cdot e^{kh}$

Since $\left.\begin{matrix} h = 0 \Rightarrow p = 1013 \\ h = 20 \Rightarrow p = 90 \end{matrix}\right\} \Rightarrow p = 1013 \cdot e^{-0.121h}$

b) $p(50) \approx 2.383$ milibars

c) $900 = 1013 \cdot e^{0.121h} \Rightarrow h \approx 0.977$ km

10. a) $\dfrac{db}{dt} = k \cdot b$

b) Integrate to obtain: $b = A \cdot e^{kt}$. Since $\left.\begin{matrix} t = 3 \Rightarrow b = 10000 \\ t = 5 \Rightarrow b = 40000 \end{matrix}\right\} \Rightarrow b = 1250 \cdot e^{0.693t}$

c) $10^{6} = 1250 \cdot e^{0.693t} \Rightarrow t \approx 9.644$ hours

d) $b = 1250 \cdot e^{0.69324} \approx 20.971$ billion. No, it does not make sense…

11. a) $\dfrac{df}{dt} = k \cdot f$

b) Integrate to obtain $f = A \cdot e^{kt}$. Since $\left.\begin{matrix} t = 0 \Rightarrow f = 370 \\ t = 10 \Rightarrow f = 900 \end{matrix}\right\} \Rightarrow f = 370 \cdot e^{0.089t}$

c) $2009 \Rightarrow t = 39 \Rightarrow f = 370 \cdot e^{(0.089)\cdot(39)} \approx \$11{,}851.158$ billions $\approx \$11.8$ trillions. Good approximation!

d) \$11,851.158 billions divided by 300 million people: \$39,503.86 was your share of our national debt!