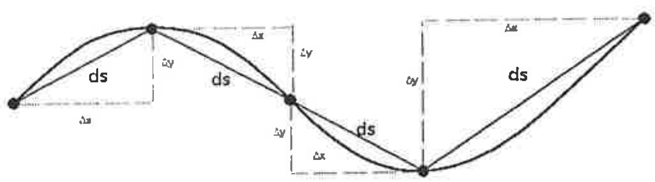
Arc Length. Surface Area.

Arc Length. Suppose that $y = f(x)$ is a continuous function with a continuous derivative on $[a, b]$. The arc length L of $f(x)$ for $a \le x \le b$ can be obtained by integrating the length element ds from a to b. The length element ds on a sufficiently small interval can be approximated by the hypotenuse of a triangle with sides dx and dy.



Thus $ds^2 = dx^2 + dy^2 \Rightarrow ds = \sqrt{dx^2 + dy^2}$ and so

$$L = \int_a^b ds = \int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{(1 + \frac{dy^2}{dx^2})dx^2} = \int_a^b \sqrt{(1 + \frac{dy^2}{dx^2})}dx.$$

Note that $\frac{dy^2}{dx^2} = \left(\frac{dy}{dx}\right)^2 = (y')^2$. So the formula for the arc length becomes

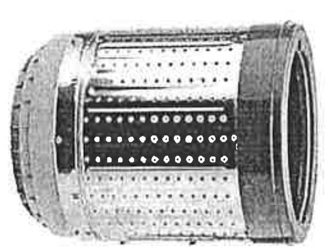
$$L = \int_a^b \sqrt{1 + (y')^2}\ dx.$$

Area of a surface of revolution. Suppose that $y = f(x)$ is a continuous function with a continuous derivative on $[a, b]$. To compute the surface area S_x of the surface obtained by rotating $f(x)$ about x-axis on $[a, b]$, we can integrate the surface area element dS which can be approximated as the product of the circumference $2\pi y$ of the circle with radius y and the height that is given by the arc length element ds. Since ds is $\sqrt{1 + (y')^2 dx}$, the formula that computes the surface area is

$$S_x = \int_a^b 2\pi y \sqrt{1 + (y')^2}\ dx.$$

If $y = f(x)$ is rotated about y-axis on $[a, b]$, then dS is the product of the circumference $2\pi x$ of the circle with radius x and the height that is given by the arc length element ds. Thus, the formula that computes the surface area is

$$S_y = \int_a^b 2\pi x \sqrt{1 + (y')^2}\ dx.$$



Practice Problems.

1. Find the length of the curve $y = x^{3/2}$, $1 \leq x \leq 4$.

2. Find the length of the curve $y = \sqrt{1 - x^2}$, $-1 \leq x \leq 1$.

3. Use the Left-Right sum calculator program to approximate the length of the curve $y = x^3$, for $0 \leq x \leq 1$ to two digits.

4. Use the Left-Right sum calculator program to approximate the length of the curve $y = \sin x$, for $0 \leq x \leq \pi$ to five digits.

5. Use the Left-Right sum calculator program with $n = 100$ subintervals to approximate the length of the curve $y = e^x$, for $0 \leq x \leq 1$.

6. Find the area of the surface obtained by rotating $y = x^3$, for $0 \leq x \leq 2$ about the x-axis.

7. Find the area of the surface obtained by rotating $y = \sqrt{x}$, for $4 \leq x \leq 9$ about the x-axis.

8. Find the area of the surface obtained by rotating $y = x^2$, for $1 \leq x \leq 2$ about the y-axis.

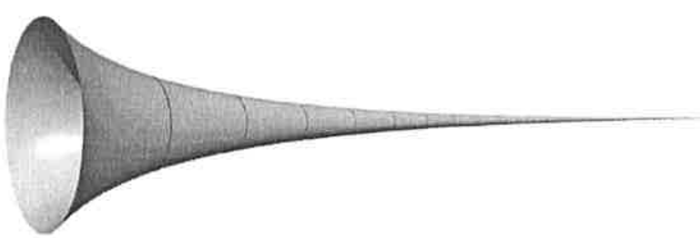
9. Prove the formula $4r^2\pi$ computes the surface area of a sphere with radius r.

10. Use the Left-Right sum calculator program to approximate the surface area obtained by rotating the curve $y = \sin x$, for $0 \leq x \leq \pi$ about x-axis to four digits.

11. Use the Left-Right sum calculator program with 100 subintervals to find the Left sum which approximates the surface area of the surface obtained by rotating $y = e^{x^2+1}$ $0 \leq x \leq 1$, about x-axis.

12. Use the Left-Right sum calculator program with 100 subintervals to find the Right sum which approximates the surface area of the surface obtained by rotating $y = \ln(x^3 + 1)$ $0 \leq x \leq 1$, about y-axis.

13. **A solid with infinite surface area that encloses a finite volume.** The surface of revolution obtained by revolving $y = \frac{1}{x}$ for $1 \leq x \leq \infty$ is known as the **Gabriel's Horn or Torricelli's trumpet**. Using the inequality $1 + \frac{1}{x^4} > 1$, demonstrate that this surface has infinite surface area. Then find the volume enclosed by this surface and show it is finite.



In Calculus 3, we will encounter another example of a similar phenomenon: a fractal object called **Koch snowflake** with infinite perimeter that encloses a finite area.

Solutions.