

**SOLUTIONS  
TO  
ORTHOGONAL  
TRAJECTORIES  
EXERCISES**

1. We start by finding an expression for slope for the given family.

$$y^2 = kx^3 \Rightarrow 2y \frac{dy}{dx} = 3kx^2 \Rightarrow \frac{dy}{dx} = \frac{3kx^2}{2y}$$

The expression for slope must be independent of the parameter  $k$ , so solving for  $k$  in the family equation we have

$$k = \frac{y^2}{x^3} \Rightarrow \frac{dy}{dx} = \frac{3kx^2}{2y} = \frac{3x^2}{2y} \cdot \frac{y^2}{x^3} = \frac{3y}{2x}$$

At a point of intersection, the slope of the orthogonal trajectory is

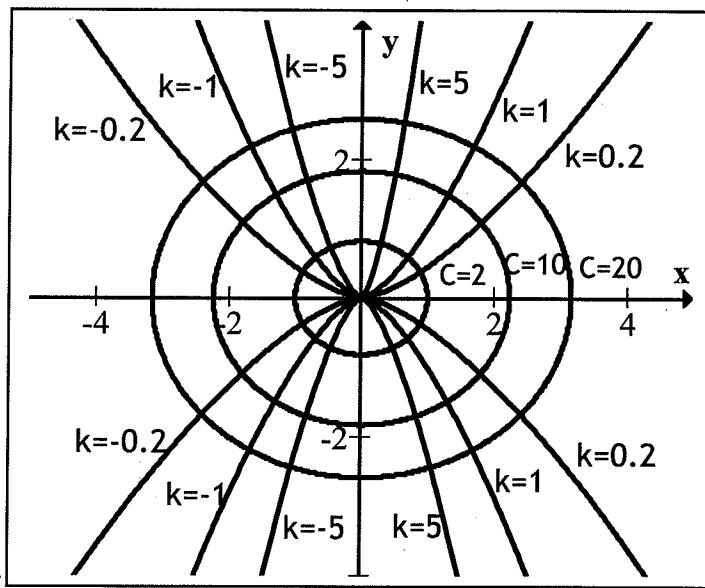
$$\frac{dy}{dx} = -\frac{1}{\left(\frac{3y}{2x}\right)} = -\frac{2x}{3y}$$

This is a separable D.E. and so

$$3y dy = -2x dx \Rightarrow \int 3y dy = \int -2x dx \Rightarrow \frac{3y^2}{2} = -x^2 + C_1 \Rightarrow 3y^2 = -2x^2 + 2C_1$$

Re-labelling  $2C_1 = C$ , the family of orthogonal trajectories is  $2x^2 + 3y^2 = C$ .

The picture looks like this.



2. We start by finding an expression for slope for the given family.

$$2x^2 - y^2 = a \Rightarrow 4x - 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-4x}{-2y} = \frac{2x}{y}$$

At a point of intersection, the slope of the orthogonal trajectory is

$$\frac{dy}{dx} = -\frac{1}{\left(\frac{2x}{y}\right)} = -\frac{y}{2x}$$

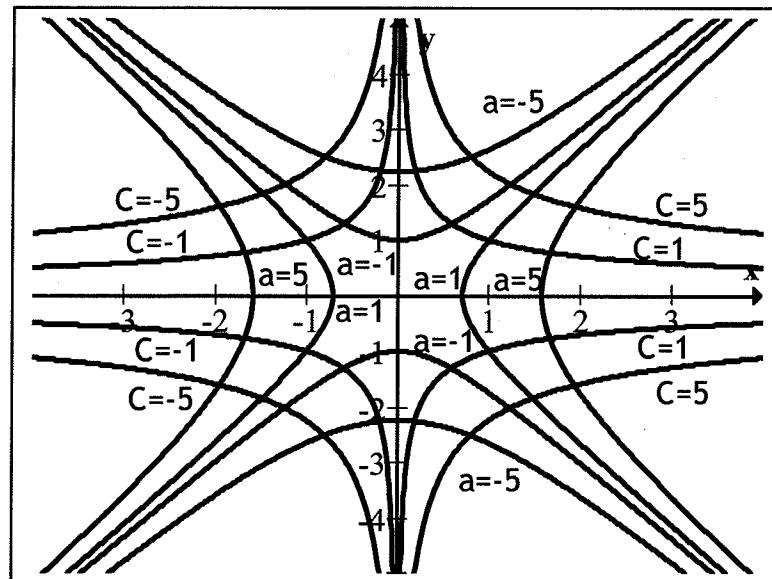
This is a separable D.E. and so

$$\begin{aligned} \frac{dy}{y} = \frac{dx}{-2x} &\Rightarrow \int \frac{1}{y} dy = -\frac{1}{2} \int \frac{1}{x} dx \Rightarrow \ln(|y|) = -\frac{1}{2} \ln(|x|) + C_1 \\ &\Rightarrow -2 \ln(|y|) = \ln(|x|) + C_1 \Rightarrow \ln(|y|)^{-2} = \ln(|x|) + C_1 \\ &\Rightarrow \ln\left(\frac{1}{y^2}\right) = \ln(|x|) + C_1 \Rightarrow \frac{1}{y^2} = e^{\ln(|x|)+C_1} \Rightarrow \frac{1}{y^2} = e^{\ln(|x|)} \cdot e^{C_1} \\ &\Rightarrow \frac{1}{y^2} = C_2 |x|, \text{ where } C_2 = e^{C_1} > 0 \\ &\Rightarrow y^2 = \frac{C}{x}, \text{ where } C \neq 0 \end{aligned}$$

Note: The D.E. is satisfied when  $y=0$ , so  $C$  can be any real number.

The family of orthogonal trajectories is  $y^2 = \frac{C}{x}$ , where  $C$  is any real number.

The picture looks like this.



3. We start by finding an expression for slope for the given family.

$$y = k e^x \Rightarrow \frac{dy}{dx} = k e^x$$

The expression for slope must be independent of the parameter  $k$ , so solving for  $k$  in the family equation we have

$$k = \frac{y}{e^x} \Rightarrow \frac{dy}{dx} = k e^x = \frac{y}{e^x} \cdot e^x = y$$

At a point of intersection, the slope of the orthogonal trajectory is  $\frac{dy}{dx} = -\frac{1}{y}$

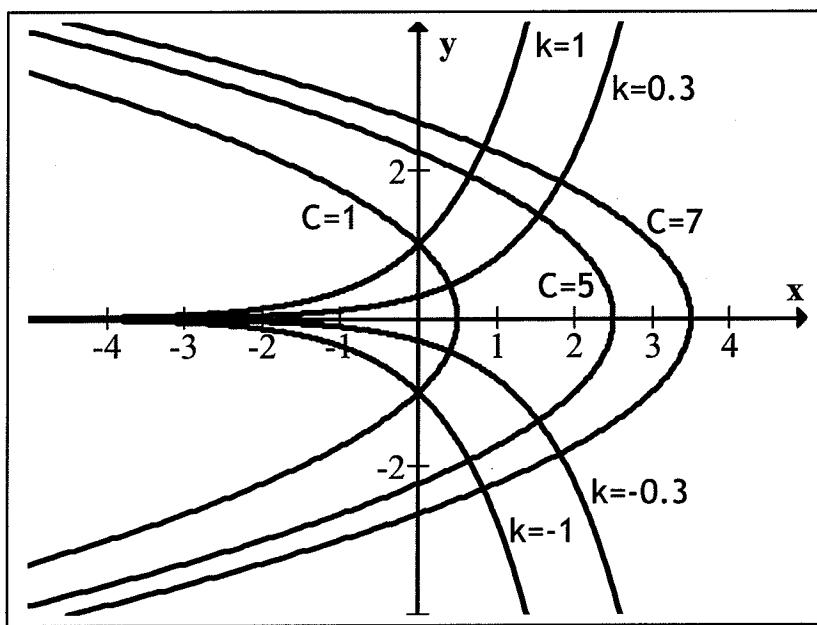
This is a separable D.E. and so

$$y dy = -dx \Rightarrow \int y dy = -\int dx \Rightarrow \frac{y^2}{2} = -x + C_1 \Rightarrow y^2 = -2x + 2C_1$$

Re-labelling  $2C_1 = C$ , the family of orthogonal trajectories is  $y^2 = -2x + C$  or

$$x = -\frac{1}{2}y^2 + B, \text{ where } B = \frac{C}{2}$$

The picture looks like this.



4. We start by finding an expression for slope for the given family.

$$y = \frac{1}{x+k} = (x+k)^{-1} \Rightarrow \frac{dy}{dx} = -(x+k)^{-2} = -\frac{1}{(x+k)^2}$$

The expression for slope must be independent of the parameter  $k$ , so we eliminate  $k$  by observing

$$y = \frac{1}{x+k} \Rightarrow \frac{dy}{dx} = -\frac{1}{(x+k)^2} = -\frac{1}{x+k} \cdot \frac{1}{x+k} = -y^2$$

At a point of intersection, the slope of the orthogonal trajectory is  $\frac{dy}{dx} = -\frac{1}{(-y^2)} = \frac{1}{y^2}$

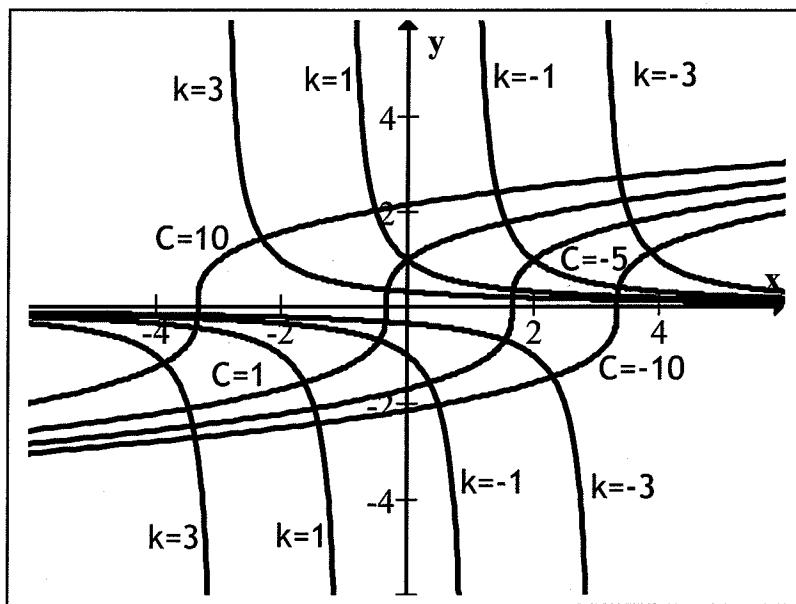
This is a separable D.E. and so

$$y^2 dy = dx \Rightarrow \int y^2 dy = \int dx \Rightarrow \frac{y^3}{3} = x + C_1 \Rightarrow y^3 = 3x + 3C_1$$

Re-labelling  $3C_1 = C$ , the family of orthogonal trajectories is  $y^3 = 3x + C$  or

$$x = \frac{1}{3}y^3 + B, \text{ where } B = -\frac{C}{3}$$

The picture looks like this.



**SOLUTIONS**

**TO**

**MIXTURE**

**PROBLEMS**

1. Let  $y(t)$  denote the amount of salt (in kg) in the tank after  $t$  min.

Notice that the solution flows out at the same rate it enters and so the volume of the solution in the tank remains constant at 2000 L.

$$\begin{aligned}\text{Rate in} &= (0.03 \text{ kg/L})(4 \text{ L/min}) + (0.05 \text{ kg/L})(8 \text{ L/min}) \\ &= 0.52 \text{ kg/min}\end{aligned}$$

$$\text{Rate out} = \left(\frac{y(t)}{2000} \text{ kg/L}\right)(12 \text{ L/min}) = \frac{3y}{500} \text{ kg/min}$$

$$\text{DE: } \frac{dy}{dt} = 0.52 - \frac{3y}{500}; \quad y(0)=0$$

$$\frac{dy}{0.52 - \frac{3}{500}y} = dt \quad \rightarrow 0.52 - \frac{3}{500}y = C_4 e^{-\frac{3t}{500}}, \quad C_4 \neq 0$$

$$\int \frac{1}{0.52 - \frac{3}{500}y} dy = \int dt$$

$$-\frac{500}{3} \int \frac{-3/500}{0.52 - \frac{3}{500}y} dy = \int dt$$

$$-\frac{500}{3} \ln |0.52 - \frac{3}{500}y| = t + C_1$$

$$\ln |0.52 - \frac{3}{500}y| = -\frac{3t}{500} + C_2$$

$$|0.52 - \frac{3}{500}y| = C_3 e^{-\frac{3t}{500}}, \quad C_3 > 0$$

$$\frac{3}{500}y = 0.52 - C_4 e^{-\frac{3t}{500}}$$

$$y = \frac{260}{3} - C e^{-\frac{3t}{500}}, \quad C \neq 0$$

$$y(0)=0 \Rightarrow 0 = \frac{260}{3} - C e^0$$

$$\Rightarrow C = \frac{260}{3}$$

$$\therefore y(t) = \frac{260}{3} \left[ 1 - e^{-\frac{3t}{500}} \right]$$

$$\text{Want: } y(90)$$

$$y(90) = \frac{260}{3} \left[ 1 - e^{-\frac{3(90)}{500}} \right] \approx 36.2$$

Conclusion: After 1.5 hours there is approximately 36.2 kg of salt in the tank.

2. Let  $y(t)$  denote the amount of toxin (in grams) in the lake after  $t$  days.

Notice that water flows in and out of the lake at the same rate. So that the volume of the lake stays fixed at  $10^6 \text{ m}^3$ .

Want:  $\lim_{t \rightarrow \infty} \frac{y(t)}{10^6}$  } concentration of toxin at time  $t$

$$\text{Rate in} = \left( \frac{5 \text{ gm}}{\text{m}^3} \right) \left( 1000 \frac{\text{m}^3}{\text{day}} \right) = 5000 \text{ gm/day}$$

$$\text{Rate out} = \left( \frac{y(t) \text{ gm}}{10^6 \text{ m}^3} \right) \left( 1000 \frac{\text{m}^3}{\text{day}} \right) = \frac{y(t)}{1000} \text{ gm/day.}$$

$$\text{D.E. : } \frac{dy}{dt} = 5000 - \frac{y(t)}{1000}; \quad y(0) = 0$$

$$\frac{dy}{5000 - \frac{y}{1000}} = dt$$

$$-\int_{-1000}^0 \frac{-\frac{1}{1000}}{5000 - \frac{y}{1000}} dy = \int dt$$

$$-1000 \ln \left| 5000 - \frac{y}{1000} \right| = t + C_1,$$

$$\ln \left| 5000 - \frac{y}{1000} \right| < -0.001t + C_2$$

$$\left| 5000 - \frac{y}{1000} \right| = C_3 e^{-0.001t}, \quad C_3 > 0$$

$$5000 - \frac{y}{1000} = C_4 e^{-0.001t}, \quad C_4 \neq 0$$

$$\begin{aligned} \frac{y}{1000} &= 5000 - C_4 e^{-0.001t} \\ y &= 5 \times 10^6 - C e^{-0.001t}, \quad C \neq 0 \\ y(0) = 0 &\Rightarrow 0 = 5 \times 10^6 - C e^0 \\ \Rightarrow C &= 5 \times 10^6 \\ \therefore y(t) &= 5 \times 10^6 \left[ 1 - e^{-0.001t} \right] \end{aligned}$$

$$\text{Now, } \lim_{t \rightarrow \infty} \frac{y(t)}{10^6} = \lim_{t \rightarrow \infty} \frac{5 \times 10^6 \left[ 1 - e^{-0.001t} \right]}{10^6}$$

$$= 5 \lim_{t \rightarrow \infty} \left[ 1 - e^{-0.001t} \right]^0 = 5 \cdot 1 = 5 \text{ g/m}^3$$

3. Let  $y(t)$  denote the amount of water (in gal) in the tank at time  $t$  min.

Let  $K$  denote the constant rate at which solution is added or drained from the tank.

Notice that since the rate of solution in is the same as out, the volume of the solution in the tank is fixed at 150 gal.

$$\text{Rate in} = K \text{ gal/min}$$

$$\text{Rate out} = \left(\frac{y}{150}\right) \cdot K \text{ gal/min}$$

$$\text{D.E.: } \frac{dy}{dt} = K - \frac{yK}{150} = \left(\frac{150-y}{150}\right)K ; y(0) = 0.7(150) = 105$$

$$\frac{\frac{dy}{(150-y)} K}{150} = dt$$

$$-\frac{150}{K} \int \frac{-1}{150-y} dy = \int dt$$

$$-\frac{150}{K} \ln|150-y| = t + C_1$$

$$\ln|150-y| = -\frac{Kt}{150} + C_2$$

$$|150-y| = C_3 e^{-\frac{Kt}{150}}, C_3 > 0$$

Find  $K$  such that  $y(70) = 150 - 30 = 120$  (gal of water)

$$120 = 150 - 45 e^{-\frac{K}{150}(70)}$$

$$45 e^{-\frac{7}{15}K} = 30$$

$$e^{-\frac{7}{15}K} = \frac{30}{45} = \frac{2}{3}$$

$$150-y = C \cdot e^{-\frac{K}{150}t}, C \neq 0$$

$$y = 150 - C e^{-\frac{K}{150}t}$$

$$y(0) = 105 :$$

$$105 = 150 - C e^0$$

$$C = 150 - 105 = 45$$

$$\therefore y(t) = 150 - 45 e^{-\frac{K}{150}t}$$

Conclusion: Pure water is poured into the tank at the rate of approximately 0.87 gal/min.

4. Let  $y(t)$  denote the amount of salt (in lbs) in the tank at time  $t$ .

Notice that the flow rates in and out are different.

Because there is a net flow out of 2 gal/min, the volume of the solution in the tank is decreasing.

At time  $t$  min,  $2t$  gal of solution have flowed out and so the volume in the tank is now  $25 - 2t$  gal.

$$\text{Rate in} = \left(0 \frac{\text{lb}}{\text{gal}}\right)(3 \text{ gal/min}) = 0 \text{ lb/min}$$

$$\text{Rate out} = \left(\frac{y(t)}{25-2t} \frac{\text{lb}}{\text{gal}}\right)(5 \text{ gal/min}) = \frac{5y}{25-2t} \text{ lb/min}$$

$$\text{D.E.: } \frac{dy}{dt} = 0 - \frac{5y}{25-2t} = -\frac{5y}{25-2t}; \quad y(0) = (0.03)(25) = 0.75$$

$$\frac{dy}{y} = -\frac{5}{25-2t} dt$$

$$\int \frac{1}{y} dy = -\frac{5}{2} \int \frac{1}{25-2t} dt$$

$$\ln|y| = \frac{5}{2} \ln|25-2t| + C_1$$

Since  $y > 0$  and  $25-2t > 0$ ,

we can drop the absolute value.

$$\ln(y) = \ln(25-2t)^{\frac{5}{2}} + C_1$$

$$y = (25-2t)^{\frac{5}{2}} \cdot e^{C_1}$$

$$y = C(25-2t)^{\frac{5}{2}}$$

$$y(0) = 0.75:$$

$$0.75 = C(25-2(0))^{\frac{5}{2}}$$

$$0.75 = C(5^5)$$

$$C = \frac{0.75}{3125} = 0.00024$$

$$\therefore y(t) = 0.00024(25-2t)^{\frac{5}{2}}$$

Problem: Find  $t$  such that  $y(t) = 0.15$

$$0.15 = 0.00024(25-2t)^{\frac{5}{2}}$$

$$(25-2t)^{\frac{5}{2}} = 625$$

$$25-2t = 625^{\frac{2}{5}}$$

$$t = \frac{25-625^{\frac{2}{5}}}{2} \approx 5.93$$

Conclusion: After about 5.93 min there will be 0.15 lb of salt in the tank.